§1. Non-Abelian Gauge Theories

§1.1 Gauge invariance

$$\mathcal{L} = \mathcal{L}(\psi, D_\mu \psi, D_\nu D_\mu \psi, \ldots, F^\alpha{}_{\mu\nu}, D_\rho F^\alpha{}_{\mu\nu} \ldots)$$

$\uparrow$

Lagrangian

ψ: matter fields

D_μ: derivatives, where $\mu = 0,1,2,3$ are the space-time directions

$F^\alpha{}_{\mu\nu}$: field strength

Assume invariance under infinitesimal trfs.

$$\delta\psi_\ell(x) = i\varepsilon^\alpha(x)\,(t_\alpha)_\ell{}^m\,\psi_m \qquad (1)$$

$\uparrow$ space-time coordinate

with some constant matrices t_α, and infinitesimal parameters $\varepsilon^\alpha(x)$

$\longrightarrow$ infinitesimal generators of Lie group:

$$[t_\alpha, t_\beta] = i\,C^\gamma{}_{\alpha\beta}\,t_\gamma \qquad (2)$$

where $C^\gamma{}_{\alpha\beta}$ are the "structure-constants" satisfying

$C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ and

$$0 = C^\delta{}_{\alpha\beta}\,C^\varepsilon{}_{\delta\gamma} + C^\delta{}_{\gamma\alpha}\,C^\varepsilon{}_{\delta\beta} + C^\delta{}_{\beta\gamma}\,C^\varepsilon{}_{\delta\alpha}$$

The last equality follows from the "Jacobi-identity" $0 = [[t_\alpha, t_\beta], t_\gamma] + [[t_\gamma, t_\alpha], t_\beta] + [[t_\beta, t_\gamma], t_\alpha]$

$\rightarrow$ adjoint representation: $(t^A_\alpha)^\beta{}_\gamma = -i C^\beta{}_{\gamma\alpha}$

Example:

i) "iso-spin":

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}$$ with t_α given by

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the commutation relations (2) with

$$C^\gamma{}_{\alpha\beta} = \varepsilon_{\gamma\alpha\beta}$$

ii) adjoint representation:

$$t^A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad t^A_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

Differentiating equation (1) gives

$$\delta(\partial_\mu \psi_\ell(x)) = i\varepsilon^\alpha(x)(t_\alpha)_\ell{}^m(\partial_\mu \psi_m(x)) + i(\partial_\mu \varepsilon^\alpha(x))(t_\alpha)_\ell{}^m \psi_m(x)$$

$\rightarrow$ to make the Lagrangian invariant, we need a field $A^\alpha{}_\mu$ with:

$$\delta A^\beta{}_\mu = \partial_\mu \varepsilon^\beta + i\varepsilon^\alpha (t^A_\alpha)^\beta{}_\gamma A^\gamma{}_\mu \qquad (3)$$

$$\longrightarrow \quad (D_\mu \psi(x))_\ell = \partial_\mu \psi_\ell(x) - i A^\beta{}_\mu(x)(t_\beta)_\ell{}^m \psi_m(x)$$

and thus

$$\delta(D_\mu \psi)_\ell = i\varepsilon^\alpha (t_\alpha)_\ell{}^m \partial_\mu \psi_m - i C^\beta{}_{\gamma\alpha} \varepsilon^\alpha A^\gamma{}_\mu (t_\beta)_\ell{}^m \psi_m$$
$$+ A^\gamma{}_\mu (t_\gamma)_\ell{}^m (t_\alpha)_m{}^n \psi_n \varepsilon^\alpha$$

using (2), we get

$$\delta(D_\mu \psi)_\ell = i\varepsilon^\alpha (t_\alpha)_\ell{}^m (D_\mu \psi)_m$$

We also notice

$$([D_\nu, D_\mu]\psi)_\ell = -i(t_\gamma)_\ell{}^m F^\gamma{}_{\nu\mu} \psi_m,$$

where $$F^\gamma{}_{\nu\mu} = \partial_\nu A^\gamma{}_\mu - \partial_\mu A^\gamma{}_\nu + C^\gamma{}_{\alpha\beta} A^\alpha{}_\nu A^\beta{}_\mu$$

$$\delta F^\beta{}_{\nu\mu} = i\varepsilon^\alpha (t^A{}_\alpha)^\beta{}_\gamma F^\gamma{}_{\nu\mu} = \varepsilon^\alpha C^\beta{}_{\gamma\alpha} F^\gamma{}_{\nu\mu} \quad (*)$$

finite transformations:

$$\psi_\ell(x) \longrightarrow \psi_{\ell\Lambda}(x) = [\exp(i t_\alpha \Lambda^\alpha(x))]_\ell{}^m \psi_m(x) \qquad (4)$$

want $$(\partial_\mu - i t_\alpha A^\alpha{}_{\mu\Lambda})\psi_\Lambda = \exp(i t_\alpha \Lambda^\alpha)(\partial_\mu - i t_\alpha A^\alpha{}_\mu)\psi,$$

thus we must impose

$$t_\alpha A^\alpha{}_{\mu\Lambda} = \exp(i t_\beta \Lambda^\beta) t_\alpha A^\alpha{}_\mu \exp(-i t_\beta \Lambda^\beta)$$
$$- i\left[\partial_\mu \exp(i t_\beta \Lambda^\beta)\right] \exp(-i t_\beta \Lambda^\beta) \qquad (5)$$

In the limit $\Lambda^\alpha(x) \longrightarrow$ inf. $\varepsilon^\alpha(x)$ equations (4) and (5) reduce to (1) and (3).

<u>Note:</u>

i) It is always possible to choose $\Lambda^\beta(x)$ so that any one spacetime component of $A^\alpha_{\mu\Lambda}(x)$ vanishes for all α everywhere:

to make $A^\alpha_{3\Lambda}(x)$ vanish, solve

$$\partial_3 \exp(it_\beta \Lambda^\beta) = -i \exp(it_\beta \Lambda^\beta)\, t_\alpha A^\alpha_3$$

ii) If $A^\alpha_{\mu\Lambda}(x)$ vanishes everywhere then so does $F^\alpha_{\mu\nu\Lambda}(x)$, but since F transforms homogeneously, $F^\alpha_{\mu\nu\Lambda}(x)$ can vanish only if $F^\alpha_{\mu\nu}(x)$ does.

$\longrightarrow$ corresponding $A^\alpha_\mu(x)$ is called "pure gauge"

§1.2 Gauge theory Lagrangians and simple Lie Groups

Our Lagrangian

$$\mathcal{L} = \mathcal{L}(\psi, D_\mu \psi, D_\nu D_\mu \psi, \dots, F^\alpha{}_{\mu\nu}, D_\rho F^\alpha{}_{\mu\nu} \dots)$$

must be gauge invariant:

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell} i (t_\alpha)_\ell{}^m \psi_m + \frac{\partial \mathcal{L}}{\partial (D_\mu \psi_\ell)} i (t_\alpha)_\ell{}^m (D_\mu \psi)_m$$

$$+ \frac{\partial \mathcal{L}}{\partial (D_\nu D_\mu \psi)_\ell} i (t_\alpha)_\ell{}^m (D_\nu D_\mu \psi)_m + \dots + \frac{\partial \mathcal{L}}{\partial F^\beta{}_{\mu\nu}} C^\beta{}_{\gamma\alpha} F^\gamma{}_{\mu\nu}$$

$$+ \frac{\partial \mathcal{L}}{\partial D_\rho F^\beta{}_{\mu\nu}} C^\beta{}_{\gamma\alpha} D_\rho F^\gamma{}_{\mu\nu} + \dots = 0 \qquad (1)$$

→ does not depend on gauge field A^α_μ explicitly, only implicity through $F_{\mu\nu}$ and D_μ
(e.e. no terms $-\frac{1}{2} m^2_{\alpha\beta} A_{\alpha\mu} A_\beta{}^\mu$)

$$\mathcal{L}_A = -\frac{1}{4} \underset{\substack{\uparrow \\ \text{constant matrix}}}{g_{\alpha\beta}} F^\alpha{}_{\mu\nu} F^{\beta\mu\nu}$$

also possible:

$$\mathcal{L}'_A = -\frac{1}{2} \underset{\substack{\uparrow \\ \text{constant matrix}}}{\theta_{\alpha\beta}} \varepsilon^{\mu\nu\rho\sigma} F^\alpha{}_{\mu\nu} F^\beta{}_{\rho\sigma} \quad \text{(breaks parity)}$$

($= d(A \wedge F)$ → derivative, does not affect Feynman rules!)

<u>Note:</u> $\mathcal{L}_A$ leads to self-interactions of the gauge field from the quadratic terms in A
$\rightarrow$ differant from electrodynamics!
(photon does not carry electric charge)

To satisfy the gauge invariance requirement, we need

$$g_{\alpha\beta} F^{\alpha}{}_{\mu\nu} C^{\beta}{}_{\gamma\delta} F^{\gamma\mu\nu} = 0 \qquad \forall\ \delta$$

$$\longrightarrow g_{\alpha\beta} C^{\beta}{}_{\gamma\delta} = - g_{\gamma\beta} C^{\beta}{}_{\alpha\delta} \qquad (2)$$

and we also have $g_{\alpha\beta}$ positive-definite.

The following are equivalent:

a) There exists a real symmetric positive-definite matrix $g_{\alpha\beta}$ that satisfies (2)

b) There is a basis $\tilde{t}_\alpha$ for the Lie algebra, for which structure constants $\tilde{C}^{\alpha}{}_{\beta\gamma}$ are fully anti-symmetric in the 3 indices.

c) The Lie algebra is a direct sum of commuting compact simple (and $U(1)$) subalgebras.